

Two parameters scaling approach to Anderson localization of weakly interacting BEC

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We numerically study the Anderson localization of weakly interacting Bose-Einstein condensate in a one-dimensional disordered potential. We show that two parameters are needed to completely describe such system, and the density profile of which can be described with the sum of two exponential functions. This is a new attempt for precise description of systems with interplay of disorder and interaction.

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I. INTRODUCTION

Disorder is ubiquitous in nature that strongly affects the properties of many physical systems even it is only a weak perturbation. Fifty years ago, the localization of individual particles or waves in a disordered crystal was predicted by Anderson [1], and thus it is called Anderson localization (AL). It can be understood as an effect of multiple reflection of a plane wave subject to random scattering or random potential barriers [2]. Later, AL has been unambiguously observed in many systems under the approximation of a single particle in a stationary disordered potential. For example, electromagnetic waves in photonic lattices with disorder [3, 4] and momentum distribution of quantum kicked rotor [5–9]. In single-particle approximation, the interaction among individual particles is not taken into consideration. However, real materials go far beyond this approximation [10, 11], and thus observing AL is difficult. This can be understood as follows. Go beyond single-particle approximation, one needs to consider the interaction among particles. When this interaction induced non-linear effect presences, the reflected waves will interfere with each other. Therefore, fully understanding the interplay of disorder and interaction is an extremely difficult task both experimentally and theoretically [11].

Ultracold quantum gas possesses unprecedented possibility of controlling almost all relevant physical parameters [12–17], and thus recognized as an ideal system for quantum simulation. A particularly interesting aspect of the system is implementing random speckle potential using laser beams [18–22], which makes this system suitable for observing AL. Recently, take advantage of highly controllability of this system, two experimental groups have reported the observation of localization of a noninteracting as well as weak interacting Bose-Einstein condensate (BEC) in two different kinds of disordered potentials [23, 24]. The final state profile is theoretically described by a single parameter called localization length (LL) [26–30]. However, when a BEC stop to expand in a disorder potential, its interaction energy is not completely converted into kinetic energy. Therefore, one can successfully describe the wing of the localized profile in terms of LL under the single-particle approximation as there only has negligi-

ble particle interaction due to long time free expansion. But this does not applicable for the center of the localized profile where interaction between particles can not be ignored. This fact naturally leads us to consider the possibility of considering the problem of describe AL in interacting system with two parameters, one for the wing and the other for the center.

In this paper, we consider a concrete example of one-dimensional (1D) BEC with repulsive interaction in a random potential. We prove that two parameters description of AL in such system is more reasonable than that of single LL parameter. This is a new method to describe the localized profile, which provides a new method for studying AL with the interplay of disorder and interaction. The rest of this paper is arranged as following. In section II, we describes 1D BEC system with repulsive interaction, and shows that there is two different LL for the wing and center parts. In section III, as the wing LL is well known, we present detail study of the center LL focusing on its scaling law. In section IV, we give an approximate analytic expression linking the density profile of atom to the two LL and discuss the deviation of the expression. Finally, a brief summary is given in section V.

II. TWO PARAMETERS DESCRIPTION OF AL

Let us consider 1D BEC with repulsive interaction initially loaded in a harmonic trapping potential $V_{ho}(z) = \frac{1}{2}M\omega^2 z^2$, where M is the atomic mass and ω is the trapping frequency. The effective 1D structure can be achieved by applying an extremely tight harmonic vertical confinement to froze the atomic motion in the other two dimensions. The atomic interaction is effectively characterized by the s -wave scattering with effective 1D interacting strength labeled as g , which is experimentally tunable using the Feshbach resonance technique [31]. Here, we consider BEC in weakly interacting regime, i.e. $\bar{n} \gg Mg/\hbar^2$, where \bar{n} is the average atomic density. Under the mean-field approximation, the dynamics of the considered system is governed by the following Gross-Pitaevskii (GP) equation

$$i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left[\frac{\hat{p}_z^2}{2M} + V_{ext}(z) + g|\psi(z, t)|^2 \right] \psi(z, t), \quad (1)$$

where \hat{p}_z is the momentum operator, μ is the chemical potential, $V_{ext}(z)$ is the whole external potential, and the normalized

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wave function $\psi(z, t)$ is corresponding to a constant total number of atoms $N = \int |\psi(z, t)|^2 dz$.

Now we look into the time-evolution of the BEC in a disordered external potential governed by Eq. (1). To this end, we follow the procedures in the experiment [24]: Firstly, prepare the 1D BEC at equilibrium of the harmonic potential without disorder, i.e., $V_{ext}(z) = V_{ho}(z)$. In the Thomas-Fermi (TF) regime ($\mu \gg \hbar\omega$), the initial wave function takes the form of an inverted parabola [13]

$$\psi(z, 0) = \sqrt{\left(\frac{\mu}{g}\right) \left(1 - \frac{z^2}{L_{TF}^2}\right)} \Theta\left(1 - \frac{|z|^2}{L_{TF}^2}\right), \quad (2)$$

where $L_{TF} = \sqrt{2\mu/\omega^2}$ is the TF half-length, and Θ denotes the Heaviside step function. Secondly, at time $t = 0$, one switches off the harmonic potential and applies a disorder potential along the expanding axis (i.e. z axis) of the BEC. We also assume that the disorder potential $V_d(z)$ is generated by laser speckle method as in the experiment [24]. It is a random potential with a truncated negative exponential single-point distribution [25]:

$$P[V(z)] = \frac{\exp[-(V(z) + V_R)/V_R]}{V_R} \Theta\left(\frac{V(z)}{V_R} + 1\right), \quad (3)$$

The average of disorder potential is set to be $\langle V_d(z) \rangle = 0$, and its correlation function $C(z) = \langle V_d(z') V_d(z' + z) \rangle = V_R^2 c(z/\sigma_R)$ with $c(u) = \sin^2(u)/u^2$ and $V_R = \sqrt{\langle V_d^2 \rangle}$ being the standard deviation. Here $C(z)$ characterizes the amplitude of the disorder fluctuation and the correlation length of the disorder. Thus the external potential after $t = 0$ is given by $V_{ext}(z) = V_d(z)$. In the case of $\sigma_R/\xi_{int} < 1$ with $\xi_{int} = \sqrt{4M\mu}$ being the initial healing length of BECs, the density profile of the BEC will take a form of an exponential-decay function when evolving enough time [24].

To investigate the AL of the system, we work out the time evolution of the atomic density profile $n(z, t) = |\psi(z, t)|^2$ with

$$\psi(z, t) = \hat{T} \exp\left(-\frac{i}{\hbar} \int_0^t H_{GP} dt\right) \psi(z, 0), \quad (4)$$

where the GP Hamiltonian is $H_{GP} = \frac{\hat{p}_z^2}{2M} + V_d(z) + g|\psi(z, t)|^2$ and \hat{T} being the time ordering operator. We numerically calculate $\psi(z, t)$ by using the standard operator-split method. According to Ref. [32], Eq. (4) can be rewritten as

$$\begin{aligned} \psi(z, t + \delta t) = & \left\{ \exp\left(-\frac{i\hat{p}_z^2}{4M\hbar}\delta t\right) \exp\left\{-\frac{i}{\hbar} [V_d(z) + g|\psi(z, t)|^2]\delta t\right\} \right. \\ & \left. \times \exp\left(-\frac{i\hat{p}_z^2}{4M\hbar}\delta t\right) + O(\delta t^3) \right\} \psi(z, t), \end{aligned} \quad (5)$$

where the high-order term $O(\delta t^3)$ comes from the non-commuting relation of the terms in H_{GP} . In the sufficiently short time step δt , this term can be safely neglected. Combining with the Fourier transform between the position and momentum spaces, we can finally get the numerical solution of

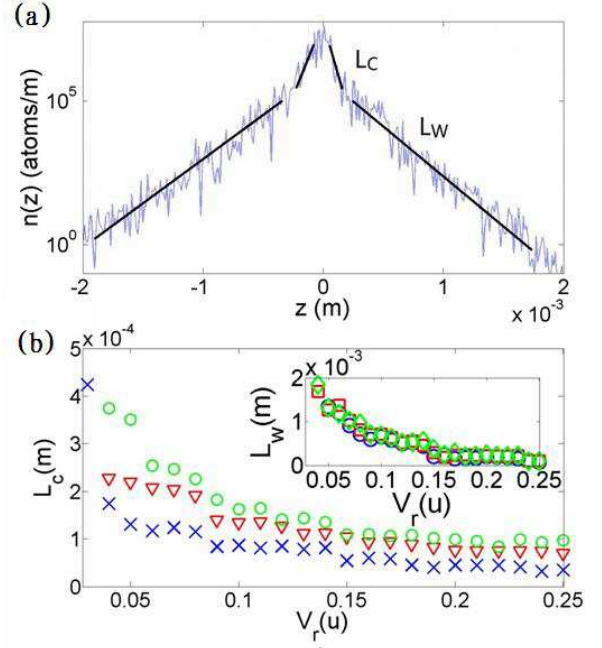


FIG. 1: (Color online) (a) The stationarity of the localized profile with disorder potential strength $V_R = 0.15\mu$, nonlinear intensity $Ng/h = 0.035$ Hz, total number of atoms $N = 1.7 \times 10^4$ and $\sigma_R/\xi_{int} \approx 0.65$. There are clearly two different LLs. (b) Central LL versus amplitude of the disordered potential in different nonlinear intensity: $Ng/h=0.035$ Hz (blue \times), 0.104 Hz (red ∇), 0.204 Hz (green \circ). Insert: Wing LL versus amplitude of the disordered potential in different nonlinear intensity: $Ng/h=0.035$ Hz (blue \circ), 0.104 Hz (red \square), 0.204 Hz (green \diamond). Both LLs are obtained in an evolution time $t = 2$ s.

$\psi(z, t)$ following the computation procedure step by step with time step δt .

As expected, as shown in Fig. 1(a), the final density profile of BEC takes the form of similar exponential function when evolving in disorder potential. We find that the wing of density profile can be exponentially fit by a wing LL denoted as L_W in Fig. 1(a), which has been considered as the *single* parameter to characterize LL of the system in experiment [24] and takes the analytic form of [12]

$$L_W = \frac{2\hbar^4}{\pi M^2 V_R^2 \sigma_R \xi_{int}^2 (1 - \sigma_R/\xi_{int})}. \quad (6)$$

However, both in our simulations and in experiment [24], the central part of the density profile can not be well-characterized by L_W though it is also localized. To describe the AL of the central part of the BEC, we introduce a center LL denoted as L_C as shown in Fig. 1(a). Here L_C is defined by the length in z axis which reduces the maximum value of the final wave function by a factor of $1/e$. Furthermore, by varying system parameters, such as disorder strength, interacting strength, etc., $L_W \neq L_C$ is always hold in our numerical simulation, as shown in Fig. 1(b). In particular, we study the effects of the nonlinear atomic interaction on the AL of the two parts, i.e. the L_W and the L_C , in varying disorder potentials. To

this end, we look into the trend of two LLs by changing the amplitude of the disordered potentials in different nonlinear interaction intensity Ng and by keeping other parameters unchanged. The results, as shown in Fig. 1(b), shows that the two LLs both nearly exponentially decrease with the increasing of the amplitude of the disordered potentials, but interestingly, they exhibit different behavior: L_W is insensitive to the nonlinear intensity as expected from Eq. (6) while L_C is strongly affected by it. This can be understood by the facts: i) for larger amplitude of the disordered potential AL is more significant, and thus LLs are both smaller; and ii) the impact of $g|\psi(z)|^2$ on the wing part L_W is slight, but they can be significant for the center part L_C since the density of this part is much larger than that of the wing part (cf. Fig. 1(a)).

Concluding this section, we have shown that it is better to use double LLs, the traditional wing one L_W and a newly defined center one L_C , to completely characterize the AL properties of the 1D weakly interacting BECs in a disorder potential. This double LLs description is quite necessary if we consider the effects of atomic interaction on the AL of the system since the two LLs have different responses on the interaction.

III. A SCALING LAW OF THE CENTER LOCALIZATION LENGTH

In contrast to the single-parameter scaling theory for the disorder system in linear regime, here we need two LLs as localization parameters to completely describe this nonlinear disorder system. The key in this nonlinear system is that the BEC finally reach nearly equilibrium in the disorder potential, and at that time the wing part will fully expand and then it can be well described with the LL for the noninteracting cases; however, the central part still contains certain residual interaction energy without fully expanding, and thus the LL of this part could be different.

To further investigate localization properties of the center part of the BEC, we numerically calculate L_C as a function of the nonlinear interaction intensity Ng with disorder potentials fixed. As shown in Fig. 2(a), we find that L_C approximately linearly depends on \sqrt{Ng} , i.e.

$$L_C \propto \sqrt{Ng} + \text{constant}, \quad (7)$$

for a disorder potential with fixed V_R . For small V_R cases, there are some variances in the fitting (cf. Fig. 2(a)). Considering the particle number N with normalization equation

$$N = \int_{-\infty}^{\infty} |\psi(z, 0)|^2 dz = \int_{-L_{TF}}^{L_{TF}} \frac{u}{g} \left(1 - \frac{z^2}{L_{TF}^2} \right) dz = \frac{4\mu L_{TF}}{3g}, \quad (8)$$

we can obtain the expression of the TF half-length as $L_{TF} = 3Ng/4\mu$, which implies that one has the relation $L_C \propto \sqrt{L_{TF}}$. Combining with the dimensional analysis, it is natural to guess that there may be a relationship between L_C and $\sqrt{L_{TF}} \times \sqrt{L_W}$. Through a simple fitting, we find that L_C has an interesting scaling law of

$$L_C \simeq \sqrt{L_W L_{TF}}, \quad (9)$$

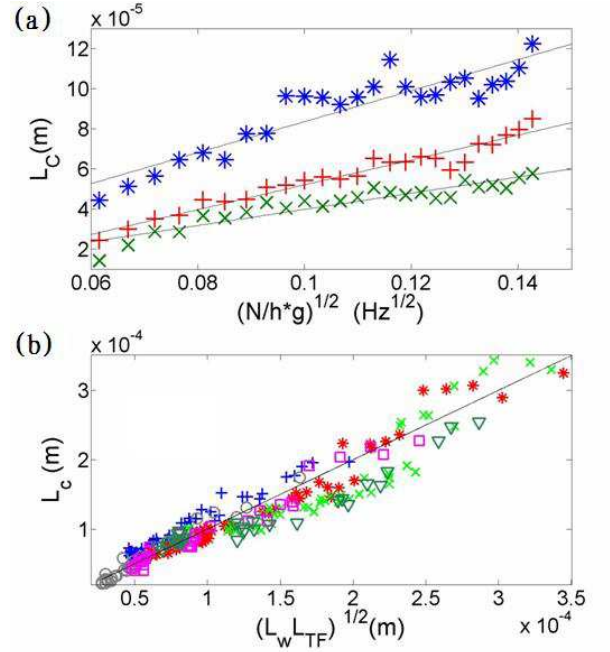


FIG. 2: (Color online) (a) The L_C versus \sqrt{g} in different V_R : $V_R=0.1u$ (blue *), $0.2u$ (red +), $0.3u$ (green \times) with black lines are theoretical predicted results by Eq. (6). (b) The L_C versus the $\sqrt{L_W L_{TF}}$. They are obtained different nonlinear intensity: $Ng/h=0.035\text{Hz}$ (blue \times), 0.104Hz (red *), 0.204Hz (green +), and different evolution time: 2.1s (gray \circ), 4.5s (purple \square), 7.0s (dark green ∇). The solid line is the function $y = x$.

as shown in Fig. 2(b).

Here we do some analysis about the physics picture of the scaling law (9). L_W can be taken as a main measurement of the strength of localization of the system, thus L_C as an additional measurement should be positively related to L_W . Since the central part of the BEC can not fully expand and has less kinetic energy, it is less sensitive to the strength of localization than the wing part does. Therefore, its power index is less than 1 and exhibits $L_C \propto \sqrt{L_W}$. Additionally, L_{TF} is a measurement of the size and the profile of the initial state, and then L_C could also be positively related to L_{TF} . Because of the disorder potential and the smaller kinetic energy, the power index is also less than 1 and exhibits $L_C \propto \sqrt{L_{TF}}$.

IV. AN APPROXIMATE ANALYTIC EXPRESSION OF DENSITY PROFILE

Now we turn to look for an analytic expression of density profile of the localized BEC using the two LLs. The previous analysis shows that its final density profile takes the form of

$$n(z, \tau) \propto \begin{cases} \exp(-2|z|/L_C) & |z| \leq z_0 \\ \exp(-2|z|/L_W) & |z| > z_0 \end{cases}, \quad (10)$$

where τ is the typical time scale when the BEC is nearly stable, and z_0 is the theoretical cross-point. Here z_0 is difficult to

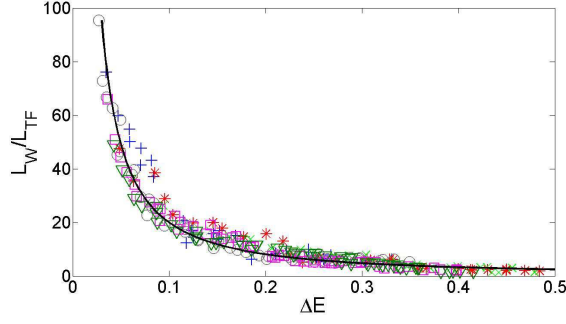


FIG. 3: (Color online) The scale parameter L_W/L_{TF} versus the residual interaction energy ΔE with $\sigma_R/\xi_{int} = 0.65$. L_W/L_{TF} is obtained from different nonlinear intensity: $Ng/h=0.035\text{Hz}$ (blue \blacklozenge), 0.104Hz (red $*$), 0.204Hz (green $+$), and different evolution time: 2.1s (gray \circ), 4.5s (purple \square), 7.0s (dark green ∇). The solid line is the best fit with the function in the Eq.(10).

determine, so we assume a simple form

$$n_a(z, \tau) = \frac{N - \Lambda}{L_W} \exp(-2|z|/L_W) + \frac{\Lambda}{L_C} \exp(-2|z|/L_C) \quad (11)$$

to describe the full density profile, where Λ is an undetermined coefficient. In the following we will see that Λ can be approximately determined.

Let us consider the normalized residual interaction energy ΔE , which is defined as

$$\begin{aligned} \Delta E &= \frac{g \int |\psi(z, \tau)|^2 dz}{g \int |\psi(z, 0)|^2 dz} \\ &= \frac{5}{6N^2} \left[\Lambda^2 \beta + (N - \Lambda)^2 \beta^2 + \frac{4(N - \Lambda)}{\beta^2 + \beta} \right], \end{aligned} \quad (12)$$

with $\beta = \sqrt{L_{TF}/L_W}$. Note that we have adopted the scaling law (9) in the derivation of Eq. (12). For the single LL cases, i.e. $\Lambda = 0$ in Eq. (12), we can find $\Delta E = 5L_{TF}/6L_W$. Based on the numerical simulation, we also find that there is a well fitting formula to characterize the relationship between ΔE and L_W/L_{TF} for our two LLs description, as shown in Fig. 3. The fitting formula is given by

$$\frac{L_W}{L_{TF}} \simeq \Delta E^{-\frac{4}{3}}. \quad (13)$$

This relation can be understood by the fact that L_W and L_{TF} are the characteristic lengths of final and initial states, and thus their ratio may have certain connection with the residual interaction energy which is the ratio of the interaction energy of final and initial states. If the BEC expands fully, the residual interaction energy tends to zero. Then, L_W will be much larger than L_{TF} , corresponding to $\Delta E \rightarrow 0$ for $L_W/L_{TF} \rightarrow \infty$ in Eq. (13) (cf. Fig. 3). On the other hand, if the expansion is relatively very small, comparable to the initial length scale of the BEC, then L_W will be nearly equivalent to L_{TF} , corresponding to the residual interaction energy tends to unit.

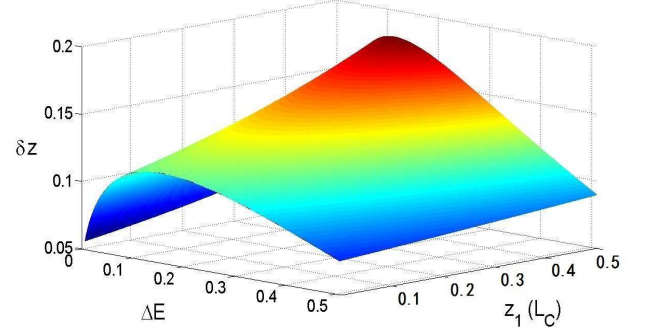


FIG. 4: (Color online) The parameter δz versus the residual interaction energy ΔE and z_1/L_C . The former is the solution of Eq. (15).

Substituting Eq. (13) into Eq. (12), one can work out Λ , which is approximately given by

$$\Lambda \simeq \left[0.75 \left(\frac{L_{TF}}{L_W} \right)^{0.1} + 0.04 \right] N \quad (14)$$

Up to this, we have shown that the full density profile of the localized BEC can be characterized by an approximate analytic function $n_a(z, \tau)$ as shown in Eq. (11), where the two LLs are used and the coefficient Λ can be determined by the equation (14).

Now we investigate the deviation in the determining L_C by using Eq. (10) and Eq. (11). To this end, we define z_1 and z_2 ($z_1, z_2 < z_0$) as the solutions of $n(z_1, \tau) = n_a(z_2, \tau)$ in the central part (i.e. $z_{1,2} < L_C$), such that we have

$$\left(\frac{N - \Lambda}{L_W} + \frac{\Lambda}{L_C} \right) e^{-2\frac{z_1}{L_C}} = \frac{N - \Lambda}{L_W} e^{-2\frac{z_2}{L_W}} + \frac{\Lambda}{L_C} e^{-2\frac{z_2}{L_C}}. \quad (15)$$

We then define the deviation δz as

$$\delta z = \frac{z_2 - z_1}{z_1}. \quad (16)$$

$\delta z > 0$ indicate the distribution of our approximation is wider than the precise distribution in the z -direction, and vice versa. According to Eqs. (13, 14, 15), we numerically compute the deviation δz as a function of ΔE and z_1 . The result in Fig. 4 shows that the deviation of z -direction with the same $n(z, t)$ and parameters. And it can find that when ΔE is in the small and large sides the deviation is smaller than that in the intermediate regime. However, the deviation using our approximation is always less than 18% in the whole regime. Relatively speaking, for non-interacting system, both the deviation of the experimental and theoretical values are larger than 50% [24]. While for the interacting system, the deviation predicted in Ref. [12] is larger than 60% with the same parameters of Fig. (1) when V_R is in the small and large sides. Therefore, we can conclude that Eq. (11) of n_a is a better approximation to describe the density profile of the localized BEC in a wide parameter range.

V. CONCLUSION

In summary, we have demonstrated that it is better and more completely to use double LLs to describe the AL of 1D weekly interacting BECs in a disordered potential. We furthermore find a scaling law related to the relationship between the newly defined LL and the nonlinear atomic interactions. An approximate analytic form of the full density profile of the localized BEC is also proposed by using the two LLs.

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